

TREE-DEPTH AND VERTEX-MINORS

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ABSTRACT. In a recent paper [6], Kwon and Oum claim that every graph of bounded rank-width is a pivot-minor of a graph of bounded tree-width (while the converse has been known true already before). We study the analogous questions for “depth” parameters of graphs, namely for the tree-depth and related new shrub-depth. We show that shrub-depth is monotone under taking vertex-minors, and that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of bounded tree-depth. We also consider the same questions for bipartite graphs and pivot-minors.

1. INTRODUCTION

Various notions of graph containment relations (e.g. graph minors) play an important part in structural graph theory. Recall that a graph H is a minor of a graph G if H can be obtained from G by a sequence of edge contractions, edge deletions and vertex deletions. In their seminal series of papers, Robertson and Seymour introduced the notion of tree-width and showed the following: The tree-width of a minor of G is at most the tree-width of G and, moreover, for each k there is a finite list of graphs such that a graph G has tree-width at most k if, and only if, no graph in the list is isomorphic to a minor of G . This, among other things, implies the existence of a polynomial-time algorithm to check that the tree-width of a graph is at most k .

There have been numerous attempts to extend this result to (or find a similar result for) “width” measures other than tree-width. The most natural candidate is clique-width, a measure generalising tree-width defined by Courcelle and Olariu [2]. However, the quest to prove a similar result for this measure has been so far unsuccessful. For one, taking the graph minor relation is clearly not sufficient as every graph on $n > 1$ vertices is a minor of the complete graph K_n , clique-width of which is 2.

However Oum [8] succeeded in finding the appropriate containment relation – called *vertex-minor* – for the notion of rank-width, which is closely related to clique-width. (More precisely, if the clique-width of a graph is k , then its rank-width is between $\log_2(k+1) - 1$ and k .) Vertex-minors are based on the operation of local complementation: taking a vertex v of a graph G we replace the subgraph induced on the neighbours of v by its edge-complement, and denote the resulting graph by $G * v$. We then say that a graph H is a vertex-minor of G if H can be obtained from G by a sequence of local complementations and vertex deletions. In [8] it was shown that if H is a vertex-minor of G , then its rank-width is at most the rank-width of G .

Another graph containment relation, the *pivot-minor*, also defined in [8], is closely related to vertex-minor. Pivot-minors are based on the operation of edge-pivoting: for an edge $e = \{u, v\}$ of a graph G we perform the operation $G * u * v * u$. Then a graph H is a pivot-minor of G if it can be obtained from G by a sequence of edge-pivotings and vertex deletions. It follows from the definition that every pivot-minor is also a vertex-minor.

This brings an interesting question: What is the exact relationship between various width measures with respect to these new graph containment relations? Recently, it was shown that every graph of rank-width k is a pivot-minor of a graph of tree-width at most $2k$ [6]. In this paper

Key words and phrases. tree-depth; shrub-depth; vertex-minor; pivot-minor.

P. Hliněný and J. Obdržálek have been supported by the Czech Science Foundation, project no. 14-03501S. S. Ordyniak have been supported by the European Social Fund and the state budget of the Czech Republic under project CZ.1.07/2.3.00/30.0009 (POSTDOC I).

O. Kwon have been supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0011653).

we investigate the existence of similar relationships for two “shallow” graph width measures: tree-depth and shrub-depth.

Tree-depth [7] is a graph invariant which intuitively measures how far is a graph from being a star. Graphs of bounded tree-depth are sparse and play a central role in the theory of graph classes of bounded expansion. *Shrub-depth* [4] is a very recent graph invariant, which was designed to fit into the gap between tree-depth and clique-width. (If we consider tree-depth to be the “shallow” counterpart of tree-width, then shrub-depth can be thought of as a “shallow” counterpart of clique-width.)

Our results can be summarised as follows. We start by showing that shrub-depth is monotone under taking vertex-minors (Corollary 3.6). Next we prove that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of bounded tree-depth (Theorem 4.4). Note that, unlike for rank-width and tree-width, restricting ourselves to pivot-minors is not sufficient. Indeed, this is because, as we prove in Proposition 4.7, graphs of bounded tree-depth cannot contain arbitrarily large cliques as pivot-minors. Interestingly, we are however able to show the same result for pivot-minors if we restrict ourselves to bipartite graphs, which were, in a similar connection, investigated already in [6]. In particular, our main result of the last section is that for any class of bounded shrub-depth there exists an integer d such that any bipartite graph in the class is a pivot-minor of a graph of tree-depth d .

2. PRELIMINARIES

In this paper, all graphs are finite, undirected and simple. A *tree* is a connected graph with no cycles, and it is *rooted* if some vertex is designated as the root. A leaf of a rooted tree is a vertex other than the root having just one neighbour. The height of a rooted tree is the maximum length of a path starting in the root (and hence ending in a leaf). Let G be a graph. We denote $V(G)$ as the vertex set of G and $E(G)$ as the edge set of G . For $v \in V(G)$, let $N_G(v)$ be the set of the neighbours of v in G .

We sometimes deal with *labelled graphs* G , which means that every vertex of G is assigned a subset (possibly empty) of a given finite label set. A graph is *m-coloured* if every vertex is assigned exactly one of given m labels (this notion has no relation to ordinary graph colouring).

We now briefly introduce the *monadic second order logic* (MSO) over graphs and the concept of FO (MSO) graph interpretation. MSO is the extension of first-order logic (FO) by quantification over sets, and comes in two flavours, MSO_1 and MSO_2 , differing by the objects we are allowed to quantify over:

Definition 2.1 (MSO_1 logic of graphs). *The language of MSO_1 consists of expressions built from the following elements:*

- *variables x, y, \dots for vertices, and X, Y for sets of vertices,*
- *the predicates $x \in X$ and $\text{edge}(x, y)$ with the standard meaning,*
- *equality for variables, quantifiers \forall and \exists ranging over vertices and vertex sets, and the standard Boolean connectives.*

MSO_1 logic can be used to express many interesting graph properties, such as 3-colourability. We also mention MSO_2 logic, which additionally includes quantification over edge sets and can express properties which are not MSO_1 definable (e.g. Hamiltonicity). The large expressive power of both MSO_1 and MSO_2 makes them a very popular choice when formulating algorithmic metatheorems (e.g., for graphs of bounded clique-width or tree-width, respectively).

The logic we will be mostly concerned with is an extension of MSO_1 called *Counting monadic second-order logic* (CMSO_1). In addition to the MSO_1 syntax CMSO_1 allows the use of predicates $\text{mod}_{a,b}(X)$, where X is a set variable. The semantics of the predicate $\text{mod}_{a,b}(X)$ is that the set X has a modulo b elements. We use C_2MSO_1 to denote the parity counting fragment of CMSO_1 , i.e. the fragment where the predicates $\text{mod}_{a,b}(X)$ are restricted to $b = 2$.

A useful tool when solving the model checking problem on a class of structures is the ability to “efficiently translate” an instance of the problem to a different class of structures, for which we already have an efficient model checking algorithm. To this end we introduce simple FO/ MSO_1

graph interpretation, which is an instance of the general concept of interpretability of logic theories [10] restricted to simple graphs with vertices represented by singletons.

Definition 2.2. A FO (MSO₁) graph interpretation is a pair $I = (\nu, \mu)$ of FO (MSO₁) formulae (with 1 and 2 free variables respectively) in the language of graphs, where μ is symmetric (i.e. $G \models \mu(x, y) \leftrightarrow \mu(y, x)$ in every graph G). To each graph G it associates a graph G^I , which is defined as follows:

- The vertex set of G^I is the set of all vertices v of G such that $G \models \nu(v)$;
- The edge set of G^I is the set of all the pairs $\{u, v\}$ of vertices of G such that $G \models \nu(u) \wedge \nu(v) \wedge \mu(u, v)$.

This definition naturally extends to the case of vertex-labelled graphs (using a finite set of labels, sometimes called colours) by introducing finitely many unary relations in the language to encode the labelling.

For example, a complete graph can be interpreted in any graph (with the same number of vertices) by letting $\nu \equiv \mu \equiv \text{true}$, and the complement of a graph has an interpretation using $\mu(x, y) \equiv \neg \text{edge}(x, y)$.

Vertex-minors and Pivot-minors. For $v \in V(G)$, the *local complementation* at a vertex v of G is the operation which complements the adjacency between every pair of two vertices in $N_G(v)$. The resulting graph is denoted by $G * v$. We say that two graphs are *locally equivalent* if one can be obtained from the other by a sequence of local complementations. For an edge $uv \in E(G)$, *pivoting* an edge uv of G is defined as $G \wedge uv = G * u * v * u = G * v * u * v$. A graph H is a *vertex-minor* of G if H is obtained from G by applying a sequence of local complementations and deletions of vertices. A graph H is a *pivot-minor* of G if H is obtained from G by applying a sequence of pivoting edges and deletions of vertices. From the definition of pivoting every pivot-minor of a graph is also its vertex-minor.

Pivot-minors of graphs are closely related to a matrix operation called pivoting. To give the exact relationship (Proposition 2.5) we will need to introduce some matrix concepts.

Pivoting on a Matrix. For two sets A and B , we denote by $A \Delta B = (A \setminus B) \cup (B \setminus A)$ its symmetric difference. Let M be a $S \times T$ matrix. For $A \subseteq S$ and $B \subseteq T$, we denote the $A \times B$ submatrix of M as $M[A, B] = (m_{i,j})_{i \in A, j \in B}$. If $A = B$, then $M[A] = M[A, A]$ and we call it a *principal submatrix* of M . If $a \in S$ and $b \in T$, then we denote $M_{a,b} = M[\{a\}, \{b\}]$. The *adjacency matrix* $A(G)$ of G is the $V(G) \times V(G)$ matrix such that for $v, w \in V(G)$, $A(G)_{v,w} = 1$ if v is adjacent to w in G , and $A(G)_{v,w} = 0$ otherwise.

Let

$$M = \begin{matrix} & \begin{matrix} S & X \setminus S \end{matrix} \\ \begin{matrix} S \\ X \setminus S \end{matrix} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}$$

be a $X \times X$ matrix over a field F .

If $A = M[S]$ is non-singular, then we define *pivoting* S on the matrix M as

$$M * S = \begin{matrix} & \begin{matrix} S & X \setminus S \end{matrix} \\ \begin{matrix} S \\ X \setminus S \end{matrix} & \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix} \end{matrix}.$$

It is sometimes called a *principal pivot transformation* [11]. The following theorem is useful when dealing with matrix pivoting.

Theorem 2.3 (Tucker [12]). *Let M be a $X \times X$ matrix over a field. If $M[S]$ is a non-singular principal submatrix of M , then for every $T \subseteq X$, $(M * S)[T]$ is non-singular if and only if $M[S \Delta T]$ is non-singular.*

Proof. See Bouchet's proof in Geelen [5, Theorem 2.7]. □

Theorem 2.4. *Let M be a $X \times X$ matrix over a field. If $M[S]$ and $(M * S)[T]$ are non-singular, then $(M * S) * T = M * (S \Delta T)$.*

Proof. See Geelen [5, Theorem 2.8]. \square

We are now ready to state the relationship between pivot-minors and matrix pivots. The proof of the following proposition uses Theorem 2.3 and Theorem 2.4, and we refer the reader to [6] for detailed explanation.

Proposition 2.5. *Graph H is a pivot-minor of G if and only if H is the graph whose adjacency matrix is $(A(G) * X)[Y]$ where $X, Y \subseteq V(G)$ and $A(G)[X]$ is non-singular.*

Tree-depth. For a forest T , the closure $\text{Clos}(T)$ of T is the graph obtained from T by making every vertex adjacent to all of its ancestors. The *tree-depth* of a graph G , denoted by $\text{td}(G)$, is one more than the minimum height of a rooted forest T such that $G \subseteq \text{Clos}(T)$.

3. SHRUB-DEPTH AND VERTEX-MINORS

In this section we show the first of our results – that shrub-depth is monotone under taking vertex-minors. The shrub-depth of a graph class is defined by the following very special kind of a simple FO interpretation:

Definition 3.1 (Tree-model [4]). *We say that a graph G has a tree-model of m colours and depth d if there exists a rooted tree T (of height d) such that:*

- i. *the set of leaves of T is exactly $V(G)$,*
- ii. *the length of each root-to-leaf path in T is exactly d ,*
- iii. *each leaf of T is assigned one of m colours (i.e. T is m -coloured),*
- iv. *and the existence of an edge between $u, v \in V(G)$ depends solely on the colours of u, v and the distance between u, v in T .*

The class of all graphs having such a tree-model is denoted by $\mathcal{TM}_m(d)$.

For example, $K_n \in \mathcal{TM}_1(1)$ or $K_{n,n} \in \mathcal{TM}_2(1)$. We thus consider:

Definition 3.2 (Shrub-depth [4]). *A class of graphs \mathcal{S} has shrub-depth d if there exists m such that $\mathcal{S} \subseteq \mathcal{TM}_m(d)$, while for all natural m it is $\mathcal{S} \not\subseteq \mathcal{TM}_m(d-1)$.*

It is easy to see that each class $\mathcal{TM}_m(d)$ is closed under complements and induced subgraphs, but neither under disjoint unions, nor under subgraphs. However, the class $\mathcal{TM}_m(d)$ is not closed under local complementations. On the other hand, to prove that shrub-depth is closed under vertex-minors it is sufficient to show that for each m there exists m' such that all graphs locally equivalent to those in $\mathcal{TM}_m(d)$ belong to $\mathcal{TM}_{m'}(d)$. As shrub-depth does not depend on m , this will be our proof strategy. Note that Definition 3.2 is asymptotic as it makes sense only for infinite graph classes; the shrub-depth of a single finite graph is always at most one. For instance, the class of all cliques has shrub-depth 1. More interestingly, graph classes of certain shrub-depth are characterisable exactly as those having simple CMSO₁ interpretations in the classes of rooted labelled trees of fixed height:

Theorem 3.3 ([4, 3]). *A class \mathcal{S} of graphs has a simple CMSO₁ interpretation in the class of all finite rooted labelled trees of height $\leq d$ if, and only if, \mathcal{S} has shrub-depth at most d .*

Proof sketch. In [4] this statement occurs with a little shift—involving MSO₁ logic instead of CMSO₁. However, since the proof in [4] builds everything on one technical claim (kernelization of MSO on trees of bounded height) which has been subsequently extended to CMSO in [3, Section 3.2], the full statement follows as well. \square

Note that the above theorem implies that any class of graphs of bounded shrubdepth is closed under simple CMSO₁ interpretations, i.e., the class of graphs obtained via a simple CMSO₁ interpretation on a class of graphs of bounded shrub-depth has itself bounded shrub-depth. This is one of the two essential ingredients we need to prove that shrub-depth is closed under vertex-minors. The other ingredient is the following technical claim:

Lemma 3.4 (Courcelle and Oum [1]). *For a graph G , let $\mathcal{L}(G)$ denote the set of graphs which are locally equivalent to G . Then there exists a simple C₂MSO₁ interpretation such that each such $\mathcal{L}(G)$ is interpreted in vertex-labellings of G .*

Proof sketch. Again, [1, Corollary 6.4] states nearly the same what we claim here. The only trouble is that [1] speaks about more general so-called transductions. Here we briefly survey that the transduction constructed in [1, Corollary 6.4] is really a simple C_2MSO_1 interpretation (we have to stay on an informal level since a formal introduction to all necessary concepts would take up several pages):

- i. In [1] local complementations of a graph G are treated via a so called isotropic system $S = S(G)$. It is, briefly, a set of $V(G)$ -indexed three-valued vectors, and so S can be described on the ground set $V(G)$ by a collection of triples of disjoint sets. This representation is definable in C_2MSO_1 [1, Proposition 6.2].
- ii. The set of graphs locally equivalent to G then corresponds to the set of isotropic systems strongly isomorphic to S . A strong isomorphism of isotropic systems on the ground set $V(G)$ is expressed in MSO_1 with respect to a suitable 6-partition of $V(G)$ by [1, Proposition 6.1].
- iii. Finally, a graph H is locally equivalent to G if and only if H is the fundamental graph of some (not unique) $S' \simeq S$ with respect to a special vector of S' , which again has a C_2MSO_1 expression with respect to a triple of subsets of $V(G)$ describing the vector (as in point i.) by [1, Proposition 6.3].

Note that all the aforementioned C_2MSO_1 expressions are on the same ground set $V(G)$. In the desired interpretation I we treat the nine parameter sets of (ii.) and (iii.) as a vertex-labelling of G , which consequently can interpret any H locally equivalent to G using C_2MSO_1 . \square

Theorem 3.5. *For a graph class \mathcal{C} , let $\mathcal{L}(\mathcal{C})$ denote the class of graphs which are locally equivalent to a member of \mathcal{C} . Then the shrub-depth of $\mathcal{L}(\mathcal{C})$ is equal to the shrub-depth of \mathcal{C} .*

Proof. Let d be the least integer such that, for some m as in Definition 3.2, it is $\mathcal{C} \subseteq \mathcal{TM}_m(d)$. Let I denote an FO interpretation of \mathcal{C} in the class \mathcal{T}_d of rooted labelled trees of height d which naturally follows from Definition 3.1, and let J be the simple C_2MSO_1 interpretation from Lemma 3.4.

For every $H \in \mathcal{L}(\mathcal{C})$ there is a suitably labelled graph $G \in \mathcal{C}$ such that $H \simeq G^J$, and a tree $T \in \mathcal{T}_d$ such that $G \simeq T^I$. As this T can additionally inherit any suitable labelling of G , we can claim $H \simeq (T^I)^J$. Therefore, the composition $J \circ I$ is a C_2MSO_1 interpretation of $\mathcal{L}(\mathcal{C})$ in \mathcal{T}_d . By Theorem 3.3, $\mathcal{L}(\mathcal{C})$ is of shrub-depth at most d and, at the same time, $\mathcal{C} \subseteq \mathcal{L}(\mathcal{C})$. \square

Corollary 3.6. *The shrub-depth parameter is monotone under taking vertex-minors over graph classes.*

Proof. By the definition, a vertex-minor is obtained as an induced subgraph of a locally equivalent graph. Since taking induced subgraphs does not change a tree-model, the claim follows from Theorem 3.5. \square

4. FROM SMALL TREE-DEPTH TO SMALL SC-DEPTH

We have just seen that taking vertex-minors does not increase the shrub-depth of a graph class. It is thus interesting to ask whether, perhaps, every class of bounded shrub-depth could be constructed by taking vertex-minors of some special graph class. This indeed turns out to be true in a very natural way—the special classes in consideration are the graphs of bounded tree-depth.

Before proceeding we need to introduce another “depth” parameter asymptotically related to shrub-depth which, unlike the former, is defined for any single graph. Let G be a graph and let $X \subseteq V(G)$. We denote by \overline{G}^X the graph G' with vertex set $V(G)$ where $x \neq y$ are adjacent in G' if either

- (i) $\{x, y\} \in E(G)$ and $\{x, y\} \not\subseteq X$, or
- (ii) $\{x, y\} \notin E(G)$ and $\{x, y\} \subseteq X$.

In other words, \overline{G}^X is the graph obtained from G by complementing the edges on X .

Definition 4.1 (SC-depth [4]). *We define inductively the class $SC(k)$ as follows:*

- i. let $SC(0) = \{K_1\}$;

- ii. if $G_1, \dots, G_p \in \mathcal{SC}(k)$ and $H = G_1 \dot{\cup} \dots \dot{\cup} G_p$ denotes the disjoint union of the G_i , then for every subset X of vertices of H we have $\overline{H}^X \in \mathcal{SC}(k+1)$.

The SC-depth of G is the minimum integer k such that $G \in \mathcal{SC}(k)$.

Proposition 4.2 ([4]). *The following are equivalent for any class of graphs \mathcal{G} :*

- there exist integers d, m such that $\mathcal{G} \subseteq \mathcal{TM}_m(d)$;
- there exists an integer k such that $\mathcal{G} \subseteq \mathcal{SC}(k)$.

From Definition 4.1, one can obtain the following claim:

Lemma 4.3. *Let k be a positive integer. If a graph G has SC-depth at most k , then G is a vertex-minor of a graph of tree-depth at most $k+1$.*

Proof. For a graph G of SC-depth k , we recursively construct a graph U and a rooted forest T such that

- i. G can be obtained from U as a vertex-minor via applying local complementations only at the vertices in $V(U) \setminus V(G)$, and
- ii. $U \subseteq \text{Clos}(T)$ and T has depth k .

If $k = 0$, then it is clear by setting $G = U = T = K_1$. We assume that $k \geq 1$.

Since G has SC-depth k , there exist a graph H and $X \subseteq V(H)$ such that $G = \overline{H}^X$ and H is the disjoint union of H_1, H_2, \dots, H_m such that each H_i has SC-depth $k-1$. By induction hypothesis, for each $1 \leq i \leq m$, H_i is a vertex-minor of a graph U_i and $U_i \in \text{Clos}(T_i)$ where the height of T_i is at most k . For each $1 \leq i \leq m$, let r_i be the root of T_i , and let T be the rooted forest obtained from the disjoint union of all T_i by adding a root r which is adjacent to all r_i . Let U be the graph obtained from the disjoint union of all U_i and $\{r\}$ by adding all edges from r to X . Validity of (ii.) is clear from the construction.

Now we check the statement (i.). By our construction of U , any local complementation in U_i has no effect on U_j for $j \neq i$, and local complementations at vertices in $V(U_i) \setminus V(H_i)$ do not change edges incident with r . Hence, by induction, we can obtain H as a vertex-minor of U and still have r adjacent precisely to $X \subseteq V(H)$. We then apply the local complementation at $r \in V(U) \setminus V(H)$, and delete $V(U) \setminus V(G)$ to obtain G . \square

This, with Proposition 4.2, now immediately gives the main conclusion:

Theorem 4.4. *For any class \mathcal{S} of bounded shrub-depth, there exists an integer d such that every graph in \mathcal{S} is a vertex-minor of a graph of tree-depth d .* \square

Comparing Theorem 4.4 with [6] one may naturally ask whether, perhaps, weaker pivot-minors could be sufficient in Theorem 4.4. Unfortunately, that is very false from the beginning. Note that all complete graphs have SC-depth 1. On the other hand, we will prove (Proposition 4.7) that graphs of bounded tree-depth cannot contain arbitrarily large cliques as pivot-minors. We need the following technical lemmas.

Lemma 4.5. *Let G be a graph and $X \subseteq V(G)$ such that $\mathbf{A}(G)[X]$ is non-singular and $|X| \geq 3$. If $u \in X$, then there exist $v, w \in X \setminus \{u\}$ such that $vw \in E(G)$.*

Proof. Let $u \in X$. Suppose that for every pair of distinct vertices $v, w \in X \setminus \{u\}$, $vw \notin E(G)$. That means $G[X]$ is isomorphic to a star with the centre u . However, the matrix $\mathbf{A}(G)[X]$ is clearly singular, and it contradicts to the assumption. \square

Lemma 4.6. *Let G be a graph and let $X \subseteq V(G)$ such that $X \neq \emptyset$ and $\mathbf{A}(G)[X]$ is non-singular. Let $s \in X$. Then G has a sequence of pairs of vertices $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_m, y_m\}$ such that*

- a) $\mathbf{A}(G) * X = \mathbf{A}(G \wedge x_1 y_1 \wedge x_2 y_2 \cdots \wedge x_m y_m)$,
- b) $(\{x_i, y_i\} : 1 \leq i \leq m)$ is a partition of X (in particular, $|X|$ is even), and
- c) $s \in \{x_m, y_m\}$.

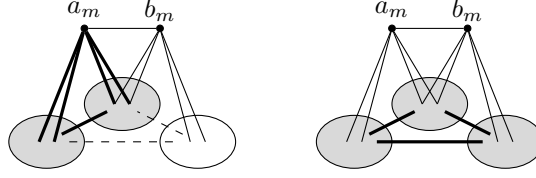


FIGURE 1. Two cases of a new clique obtained from G' by pivoting the edge $a_m b_m$ in Proposition 4.7 where $r \in \{a_m, b_m\}$. By induction hypothesis, the each coloured part can have a clique of size at most 3^{d-2} in $G' \setminus r$, and therefore the size of a new clique cannot exceed 3^{d-1} .

Proof. We prove the theorem by induction on $|X| \geq 1$. If $|X| = 1$, then $\mathbf{A}(G)[X]$ cannot be non-singular, as we have no loops in G . If $X = \{x_1, x_2\}$, then x_1, x_2 must form an edge of G since, again, $\mathbf{A}(G)[X]$ is non-singular. Since $\mathbf{A}(G) * \{x_1, x_2\} = \mathbf{A}(G \wedge x_1 x_2)$, and either $s = x_1$ or $s = x_2$, we conclude the claim.

For an inductive step, we assume that $|X| \geq 3$. Since $\mathbf{A}(G)[X]$ is non-singular, by Lemma 4.5, there exist two vertices $x_1, y_1 \in X \setminus \{s\}$ such that $x_1 y_1 \in E(G)$. Also, by Theorem 2.3, $\mathbf{A}(G \wedge x_1 y_1)[X \setminus \{x_1, y_1\}]$ is non-singular. By Theorem 2.4, we have

$$\begin{aligned} \mathbf{A}(G) * X &= \mathbf{A}(G) * (\{x_1, y_1\} \Delta (X \setminus \{x_1, y_1\})) \\ &= (\mathbf{A}(G) * \{x_1, y_1\}) * (X \setminus \{x_1, y_1\}) \\ &= \mathbf{A}(G \wedge x_1 y_1) * (X \setminus \{x_1, y_1\}). \end{aligned}$$

Since $s \in X \setminus \{x_1, y_1\} \neq \emptyset$, by the induction hypothesis, $G \wedge x_1 y_1$ has a sequence of pairs of vertices $\{x_2, y_2\}, \dots, \{x_m, y_m\}$ such that

- a) $\mathbf{A}(G \wedge x_1 y_1) * (X \setminus \{x_1, y_1\}) = \mathbf{A}((G \wedge x_1 y_1) \wedge x_2 y_2 \cdots \wedge x_m y_m)$,
- b) $(\{x_i, y_i\} : 2 \leq i \leq m)$ is a partition of $X \setminus \{x_1, y_1\}$, and
- c) $s \in \{x_m, y_m\}$.

Thus, $\mathbf{A}(G) * X = \mathbf{A}(G \wedge x_1 y_1 \wedge x_2 y_2 \cdots \wedge x_m y_m)$ and we can easily verify that $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_m, y_m\}$ is the desired sequence. \square

Now we are ready to prove the promised negative proposition.

Proposition 4.7. *Let d, t be positive integers such that $t > 3^{d-1}$. Then a graph of tree-depth at most d cannot contain a pivot-minor isomorphic to the clique K_t .*

Proof. Let $K(d) = \max \{q : \text{td}(G) \leq d \text{ and } G \text{ has a pivot-minor isomorphic to } K_q\}$. The statement is equivalent to $K(d) \leq 3^{d-1}$. If $d = 1$, then each component of a graph of tree-depth 1 has one vertex and we have $K(1) = 1$. We assume $d \geq 2$.

We choose minimal d such that a graph G of tree-depth at most d has a pivot-minor isomorphic to K_t where $t > 3^{d-1}$. Let T be a tree-depth decomposition for G of height at most d . Since G is without loss of generality connected, T has a unique root r which is a vertex of G , too. Since G has a pivot-minor isomorphic to K_t , there exists $X \subseteq V(G)$ and $S \subseteq V(G)$ such that

- a) $\mathbf{A}(G)[X]$ is non-singular, and
- b) the graph whose adjacency matrix is $(\mathbf{A}(G) * X)[S]$ is isomorphic to K_t .

By Lemma 4.6, for $s = r$ if $r \in X$ or $s \in X$ chosen arbitrarily otherwise, there exists a sequence of pairs of vertices $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_m, b_m\}$ in G such that $\mathbf{A}(G) * X = \mathbf{A}(G \wedge a_1 b_1 \wedge a_2 b_2 \cdots \wedge a_m b_m)$ and $r \notin \{a_i, b_i\}$ for $1 \leq i \leq m-1$.

Let $G' = G \wedge a_1 b_1 \wedge a_2 b_2 \cdots \wedge a_{m-1} b_{m-1}$. Then $(G' \wedge a_m b_m)[S]$ is isomorphic to K_t , and there are two cases:

- i. $r \notin \{a_m, b_m\}$, which means that $G \setminus r$ has the pivot-minor $(G' \wedge a_m b_m) \setminus r$ containing a K_{t-1} -subgraph. Since the tree-depth of $G \setminus r$ is $t-1$ as witnessed by the decomposition $T \setminus r$, and $t-1 \geq 3^{d-1} > 3^{d-1-1}$, this contradicts our minimal choice of d .

- ii. $r = a_m$, up to symmetry. After the pivot $a_m b_m$, a new clique K in G (which is not present in G') is created in two possible ways: K belongs to the closed neighbourhood of one of a_m, b_m , or K is formed in the union of the neighbourhoods of a_m, b_m (excluding a_m, b_m). See Figure 1. In either case, K is formed on two or three, respectively, cliques of $G' \setminus \{a_m, b_m\}$. Again, by minimality of d , the largest clique contained in $G' \setminus r$ can be of size 3^{d-1-1} . Therefore, $t \leq \max(1 + 2 \cdot 3^{d-2}, 3 \cdot 3^{d-2}) = 3^{d-1}$, a contradiction.

Indeed, $t = K(d) \leq 3^{d-1}$ as desired. \square

5. BIPARTITE GRAPHS OF SMALL BSC-DEPTH

In the previous section we have seen that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of tree-depth d for some d . Moreover, we have also proved that this statement does not hold if we replace vertex-minors with pivot-minors. However this raises a question whether there is some simple condition on the graph class in question which would guarantee us the theorem to hold for pivot-minors. It turns out that one such simple restriction is to consider just bipartite graphs of bounded shrub-depth, as stated by Theorem 5.4.

To get our result, we introduce the following “depth” definition better suited to the pivot-minor operation, which builds upon the idea of SC-depth. Let G be a graph and let $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$. We denote by $\overline{G}^{(X,Y)}$ the graph G' with vertex set $V(G)$ and edge set $E(G') = E(G) \Delta \{xy : x \in X, y \in Y\}$. In other words, $\overline{G}^{(X,Y)}$ is the graph obtained from G by complementing the edges between X and Y .

Definition 5.1 (BSC-depth). *We define inductively the class $\mathcal{BSC}(k)$ as follows:*

- i. let $\mathcal{BSC}(0) = \{K_1\}$;
- ii. if $G_1, \dots, G_p \in \mathcal{BSC}(k)$ and $H = G_1 \dot{\cup} \dots \dot{\cup} G_p$, then for every pair of disjoint subsets $X, Y \subseteq V(H)$ we have $\overline{H}^{(X,Y)} \in \mathcal{BSC}(k+1)$.

The BSC-depth of G is the minimum integer k such that $G \in \mathcal{BSC}(k)$.

In general, graphs of bounded SC-depth may have arbitrarily large BSC-depth, but the two notions are anyway closely related, as in Lemma 5.2. Here $\chi(G)$ denotes the chromatic number of a graph.

- Lemma 5.2.**
- a) The BSC-depth of any graph G is at least $\lceil \log_2 \chi(G) \rceil$.
 - b) The SC-depth of G is not larger than three times its BSC-depth.
 - c) If G is bipartite, then the BSC-depth of G is not larger than its SC-depth.

Proof. a) If $H' = \overline{H}^{(X,Y)}$, then $\chi(H') \leq 2\chi(H)$ since one may use a fresh set of colours for the vertices in Y . Then the claim follows by induction from Definition 5.1.

b) We have

$$\overline{H}^{(X,Y)} = \overline{\left(\overline{H^X}^Y\right)^{X \cup Y}}$$

and so the claim directly follows by comparing Definitions 5.1 and 4.1.

c) Let $G \in \mathcal{SC}(k)$. Let $V(G) = A \cup B$ be a bipartition of G , i.e., that A and B are disjoint independent sets. We use here for G the same “decomposition” as in Definition 4.1; just replacing at every step a single set X with the pair $(X \cap A, X \cap B)$ (point ii. of the definitions). The resulting graph $G' \in \mathcal{BSC}(k)$ then fulfils the following: both A, B are independent sets in G' , and every $uv \in A \times B$ is an edge in G' if and only if uv is an edge of G . Therefore, $G = G' \in \mathcal{BSC}(k)$. \square

In particular, following Lemma 5.2 a), the BSC-depth of the clique K_n equals $\lceil \log_2 n \rceil$, while $K_{m,n}$ always have BSC-depth 1.

Lemma 5.3. *Let k be a positive integer. If a graph G is of BSC-depth at most k , then G is a pivot-minor of a graph of tree-depth at most $2k+1$.*

Proof. The proof follows along the same line as the proof of Lemma 4.3. For a graph G of BSC-depth k , we recursively construct a graph U and a rooted forest T such that

- i. G can be obtained from U as a pivot-minor via pivoting edges only between vertices in $V(U) \setminus V(G)$, and
- ii. $U \subseteq \text{Clos}(T)$ and T has depth at most $2k + 1$.

If $k = 0$, then it is clear by setting $G = U = T = K_1$. We assume that $k \geq 1$.

Since G has BSC-depth k , there exist a graph H and disjoint subsets $X, Y \subseteq V(H)$ such that $G = \overline{H}^{(X,Y)}$ and H is the disjoint union of H_1, H_2, \dots, H_m such that each H_i has BSC-depth $k - 1$. By induction hypothesis, for each $1 \leq i \leq m$, H_i is a pivot-minor of a graph U_i and $U_i \in \text{Clos}(T_i)$ where the height of T_i is at most $2(k - 1) + 1$. For each $1 \leq i \leq m$, let r_i be the root of T_i , and let T be the rooted forest obtained from the disjoint union of all T_i by adding an edge between two new vertices r_x and r_y and by connecting r_y to all r_i . Let U be the graph obtained from the disjoint union of all U_i and the vertices $\{r_x, r_y\}$ by adding an edge between r_x and r_y and all edges from r_x to X as well as all edges from r_y to Y . Validity of (ii.) is clear from the construction.

Now we check the statement (i.). By our construction of U , any pivoting on edges in U_i has no effect on U_j for $j \neq i$, and pivoting on edges in $V(U_i) \setminus V(H_i)$ does not change edges incident with r_x or r_y . Hence, by induction, we can obtain H as a pivot-minor of U and still have r_x adjacent precisely to r_y and $X \subseteq V(H)$ and r_y adjacent to r_x and $Y \subseteq V(H)$. We then pivot the edge $\{r_x, r_y\} \in V(U) \setminus V(H)$, and delete $V(U) \setminus V(G)$ to obtain G . \square

The main result of this section now immediately follows from Lemmas 5.3, 5.2 c) and Proposition 4.2.

Theorem 5.4. *For any class \mathcal{S} of bounded shrub-depth, there exists an integer d such that every bipartite graph in \mathcal{S} is a pivot-minor of a graph of tree-depth d .*

6. CONCLUSIONS

We finish the paper with two questions that naturally arise from our investigations. While the first question has a short negative answer, the second one is left as an open problem.

A *cograph* is a graph obtained from singleton vertices by repeated operations of disjoint union and (full) complementation. This well-studied concept has been extended to so called “ m -partite cographs” in [4] (we skip the technical definition here for simplicity); where cographs are obtained for $m = 1$. It has been shown in [4] that m -partite cographs present an intermediate step between classes of bounded shrub-depth and those of bounded clique-width.

The first question is whether some of our results can be extended from classes of bounded shrub-depth to those of m -partite cographs. We know that shrub-depth is monotone under taking vertex-minors (Corollary 3.6) and an analogous claim is asymptotically true also for clique-width [9]. However, the main obstacle to such an extension is the fact that m -partite cographs do not behave well with respect to local and pivot equivalence of graphs. To show this we will employ the following proposition:

Proposition 6.1 ([4]). *A path of length n is an m -partite cograph if, and only if, $n < 3(2^m - 1)$.*

By the proposition, to negatively answer our question it is enough to find a class of m -partite cographs containing long paths as pivot-minors:

Proposition 6.2. *Let H_n denote the graph on $2n$ vertices from Figure 2. Then H_n is a cograph for each $n \geq 1$, and H_n contains a path of length n as a pivot-minor.*

Proof. It is $V(H_n) = \{a_i, b_i : i = 1, 2, \dots, n\}$ and $E(H_n) = \{b_i b_j : 1 \leq i < j \leq n\} \cup \{b_i a_j : 1 \leq i \leq j \leq n\}$. The graph H_n can be constructed iteratively as follows, for $j = n, n - 1, \dots, 1$: add a new vertex a_j , complement all the edges of the graph, add a new vertex b_j , complement again. Consequently, H_n is a cograph (and, in fact, a so called threshold graph).

For the second part, we let inductively $G_1 := H_n$ and $G_j := G_{j-1} \wedge a_j b_j$ for $j = 2, \dots, n - 1$. Then, by the definition, G_2 is obtained from H_n by removing all the edges incident with b_1 except $b_1 a_2, b_1 b_2$. In particular, $G_2 \setminus \{a_1, b_1\}$ is isomorphic to H_{n-1} , and a_3, b_3 are adjacent in G_2 only to vertices other than a_1, b_1 . Consequently, by induction, G_j is obtained from G_{j-1}

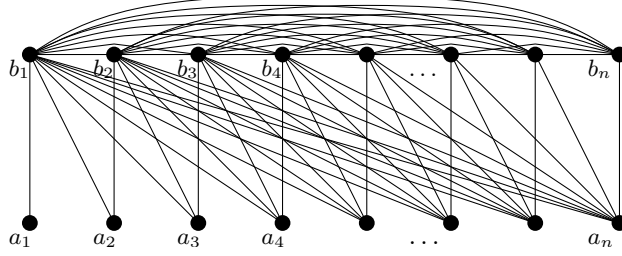


FIGURE 2. A graph on $2n$ vertices [4] which is a cograph and pivoting on $a_2b_2, a_3b_3, \dots, a_{n-1}b_{n-1}$ results in an induced path on $a_1, b_1, b_2, \dots, b_n$.

by removing all the edges incident with b_{j-1} except $b_{j-1}a_j, b_{j-1}b_j$, and G_{n-1} has the edge set $\{b_1b_2, b_2b_3, \dots, b_{n-1}b_n\} \cup \{a_1b_1, a_2b_1, a_2b_2, a_3b_2, \dots, a_nb_n\}$. Then $G_{n-1}[a_1, b_1, b_2, \dots, b_n]$ is a path. \square

Building on this negative result, it is only natural to ask whether not having a long path as vertex-minor is the property exactly characterising shrub-depth.

Conjecture 6.3. *A class \mathcal{C} of graphs is of bounded shrub-depth if, and only if, there exists an integer t such that no graph $G \in \mathcal{C}$ contains a path of length t as a vertex-minor.*

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